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# Global Existence in $L^1$ for the Enskog Equation and Convergence of the Solutions to Solutions of the Boltzmann Equation

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For the Enskog equation in a box an existence theorem is proved for initial data with finite mass, energy, and entropy. Then, by letting the diameter of the molecules go to zero, the weak convergence of solutions of the Enskog equation to solutions of the Boltzmann equation is proved.

**KEY WORDS:** Boltzmann equation; Enskog equation; kinetic theory.

## 1. INTRODUCTION

The theory of existence of solutions of the initial value problem for the Boltzmann equation recently underwent an important change when DiPerna and Lions[1] provided their ingenious proof of existence. Before, a general existence theorem was available only in the Loeb $L^1$ frame of non-standard analysis[2] (actually, the Loeb $L^1$ solutions are equivalent to standard Young measure solutions). The situation was slightly better for the Enskog equation, especially in the case of data depending on one or two space variables.[3,4] After the DiPerna and Lions result, the situation appears to be reversed. In order to reestablish a sort of equilibrium, one should now prove an existence theorem for the Enskog equation with general $L^1$ data. Recently some progress has been achieved in this direction: well-posedness and regularity for small data globally[5] as well as for large data globally[6] was obtained in $\mathbf{R}^3$. Arkeryd's proof[6] requires the initial data to possess finite moments of any order $r$ in the velocity variable,

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but also delivers uniqueness. In a previous paper,[7] an existence theorem was proved under less stringent requirements on the initial data $f_0$ but with the (unphysical) assumption, first introduced in ref. 4, that the collision kernel is symmetrized; in other words, the integral over the direction of the line joining the centers of two colliding particles is extended to the entire unit sphere rather than to a hemisphere. The proof utilized the same basic ingredients as ref. 1, but the detailed analysis of the collision operator was substituted by the use of an (equivalent) iterated integral form of the equation, giving a considerably shorter proof. The domain was assumed to be a box. At about the same time, Polewczak[8] sketched a proof for the unsymmetrized Enskog equation in $\mathbf{R}^3$; he introduced, however, an assumption on the high-density factor $\kappa$ in the Enskog equation, which essentially amounts to having a collision term dominated by a linear operator. In this paper we show how to remove the restrictions of refs. 7 and 8 and prove a global existence theorem in $L^1$ for the Enskog equation in a periodic box for the unsymmetrized case with a constant $\kappa$. The proof goes through for initial values $f_0$ with $(1+v^2+|\log f_0|)\, f_0$ in $L^1_+$ and can be easily extended to the case of $\mathbf{R}^3$ (with the additional assumption that $x^2 f_0 \in L^1$).

The second result of the paper concerns the question of the asymptotic equivalence of the Enskog and Boltzmann equations when the diameter $\sigma$ of the particles tends to zero. This matter was previously investigated in the case of smooth data with small norm[9] for various forms of the Enskog equation and for the case of general data with finite mass, energy, and entropy and a symmetrized collision term.[7] Here the latter restriction is removed.

## 2. BASIC EQUATIONS AND ENTROPY INEQUALITY

We consider the Enskog equation (EE)

$$(\partial_t + v\partial_x)\, f = Q(f) \tag{2.1}$$

with the collision operator

$$Q(f) = \int_{\mathscr{S}_+ \times \mathbf{R}^3} (f'f'_-\kappa_- - ff_+\kappa_+)\, B(v, v_*, u)\, dv_*\, du \tag{2.2}$$

where $u$ varies on the hemisphere $\mathscr{S}_+$ $(u\colon |u| = 1,\ (v - v_*)\cdot u \geqslant 0)$ and (apart from a constant factor that will be set equal to unity without loss of generality)

$$B(v, v_*, u) = \sigma^2 \max((v - v_*, u), 0) \tag{2.3}$$

The arguments of $f'$, $f'_-$, $f$, and $f_+$ are $(x, v')$, $(x-\sigma u, v'_*)$, $(x, v)$, and $(x+\sigma u, v_*)$, where

$$v' = v - u(v - v_*, u), \qquad v'_* = v_* + u(v - v_*, u) \tag{2.4}$$

The high-density factors $\kappa_\pm$ will be here taken to be equal and constant. Extending the proofs to bounded, differentiable, symmetric $\kappa$'s seems much more difficult here than in the previous paper.[7]

Equation (2.1) will be considered in a periodic box $\Lambda$, which, after rescaling, can be taken to be $\mathbf{R}^3/\mathbf{Z}^3$, with initial data $f_0 = f(0)$ such that

$$f_0(1 + v^2 + \log f_0) \in L^1_+(\Lambda \times \mathbf{R}^3) \tag{2.5}$$

The extension to $\mathbf{R}^3$ only requires adding the assumption $x^2 f_0 \in L^1$.

The key point of the proof will be the use of a modified form of the $H$-theorem valid for the Enskog gas; the first example of a theorem of this kind appears, to the best of our knowledge, in a paper by Résibois.[10] In fact, we shall introduce a modified $H$-functional, inspired by Résibois' paper[10,11]; this functional has been repeatedly used by one of the authors (C. C.) in private conversations in the last few years and a variant of it was used in the discussion of the validity of the Boltzmann equation for soft spheres.[12] The functional turns out to be the same as a functional used by Polewczak in his paper[8] (in the particular case of a constant $\kappa$). It is, however, the further rearrangement explained below that makes this functional useful in the existence proof.

Let us start now from the collision operator

$$Q(f) = \sigma^2 \kappa \int_{\mathscr{S}_+ \times \mathbf{R}^3} (f' f'_- - f f_+)(v - v_*) \cdot u \, dv_* \, du \tag{2.6}$$

and the definitions of particle density and flow

$$\rho = \int f \, dv; \qquad j = \int v f \, dv \tag{2.7}$$

Using the elementary inequality $g(\log g - \log h) \geqslant g - h$, we obtain, for a sufficiently regular $f$,

$$\int_{\mathbf{R}^3 \times \Lambda} Q(f) \log f \, dv \, dx$$

$$\leqslant \frac{1}{2} \sigma^2 \kappa \int_{\mathscr{S}_+ \times \mathbf{R}^3 \times \mathbf{R}^3 \times \Lambda} (f f_- - f f_+)(v - v_*) \cdot u \, du \, dv_* \, dv \, dx$$

$$= \frac{1}{2}\kappa\sigma^2 \int_{\mathscr{S} \times \mathbf{R}^3 \times \mathbf{R}^3 \times \Lambda} ff_-(v - v_*) \cdot u \, du \, dv_* \, dv \, dx$$

$$= \frac{1}{2}\kappa\sigma^2 \int_{\mathscr{S} \times \Lambda} (j\rho_- - j_-\rho) \cdot u \, du \, dx$$

$$= \frac{1}{2}\kappa\sigma^2 \int_{\mathscr{S} \times \Lambda} (j_+\rho - j_-\rho) \cdot u \, du \, dx$$

$$= \kappa\sigma^2 \int_{\mathscr{S} \times \Lambda} \rho j_+ \cdot u \, du \, dx$$

$$= \kappa\sigma^2 \int_{\mathscr{B}_\sigma \times \Lambda} \rho \operatorname{div} j \, dy \, dx$$

$$= -\kappa\sigma^2\sigma^2 \int_{\mathscr{B}^\sigma \times \Lambda} \frac{\partial \hat{\rho}}{\partial t} \rho \, dy \, dx$$

$$= -\frac{1}{2}\kappa\sigma^2 \frac{d}{dt} \int_{\mathscr{B}_\sigma \times \Lambda} \hat{\rho}\rho \, dy \, dx \tag{2.8}$$

Here $\mathscr{B}_\sigma$ is the ball $|x - y| \leqslant \sigma$ and a caret denotes that the argument is $y$ rather than $x$.

Hence

$$H = H_B + \frac{1}{2}\kappa\sigma^2 \int_{B_\sigma \times \Lambda} \hat{\rho}\rho \, dy \, dx$$

if $H_B$ is the usual $H$ functional, is a decreasing quantity. The additional term is not larger than $\frac{1}{2}\kappa\sigma^2 \|f\|^2_{L^1}$ and nonnegative, hence *a priori* bounded from above and below. This implies that $H_B$ is *a priori* bounded in terms of the initial data at any time, if $f$ is regular enough and Eq. (2.1) holds.

We further remark that

$$\frac{dH}{dt} = -\frac{\kappa}{2} \int_{\mathscr{S}_+ \times \mathbf{R}^3 \times \mathbf{R}^3 \times \Lambda} f'f'_-\, l\left(\frac{f'f'_-}{ff_-}\right) B \, du \, dv_* \, dv \, dx \leqslant 0 \tag{2.9}$$

where

$$l(x) = \log(x) + \frac{1 - x}{x} \geqslant 0 \tag{2.10}$$

Please note that $l(x)$ is decreasing for $x < 1$ and increasing for $x > 1$.

## 3. APPROXIMATING SOLUTIONS AND EQUIVALENT SOLUTION CONCEPTS

In order to construct the desired solutions of Eq. (2.1) a well-adapted approximation scheme is needed which retains the essential structure of the $H$-theorem for the EE discussed in Section 2.

Let $\chi$ be a decreasing $C^\infty$-function on $\mathbf{R}$ with

$$\chi(r)=1 \quad \text{for} \quad r\leqslant 1, \qquad \chi(r)=0 \quad \text{for} \quad r\geqslant 2 \tag{3.1}$$

and set

$$\chi_n(v, v_*)=\chi((v^2+v_*^2)/n^2) \tag{3.2}$$

$$Q^n(f)(x, v, t)=\int_{\mathscr{S}_+\times\mathbf{R}^3}\{\kappa\chi_n(v, v_*)\,f'f'_- - [\kappa\chi_n(v, v_*)+\varepsilon]\,ff_+\}\times B(v, v_*, u)\,dv_*\,du \tag{3.3}$$

$$(\partial_t+v\partial_x)\,f=Q^n(f) \tag{3.4}$$

with an initial value $f_0$ satisfying (2.5). We remark that $f$ depends on $n$ and we should accordingly write $f^n$ rather than $f$; we avoid, however, this complication in the notation. Now, set

$$M=\Lambda\times\mathbf{R}^3\times\mathbf{R}^3\times\mathscr{S}_+, \qquad d\mu=dx\,dv\,dv_*\,du \tag{3.5}$$

$$f^{\#}(x, v, t)=f(x+vt, v, t) \tag{3.6}$$

If we assume that the solution $f$ is nonnegative, then formal integration of Eq. (3.4) gives

$$\int_{\Lambda\times\mathbf{R}^3} f(t)\,dx\,dv=\int_{\Lambda\times\mathbf{R}^3} f_0\,dx\,dv-\varepsilon\int_0^t\int_M ff_+B\,ds\,d\mu \tag{3.7}$$

$$\int_{\Lambda\times\mathbf{R}^3} v^2 f(t)\,dx\,dv=\int_{\Lambda\times\mathbf{R}^3} v^2 f_0\,dx\,dv-\frac{\varepsilon}{2}\int_0^t\int_M (v^2+v_*^2)\,ff_+B\,ds\,d\mu \tag{3.8}$$

$$\int_{\Lambda\times\mathbf{R}^3} f(t)\log f(t)\,dx\,dv=\int_{\Lambda\times\mathbf{R}^3} f_0\log f_0\,dx\,dv+\int_0^t\int_{\Lambda\times\mathbf{R}^3}\partial_s f^{\#}(s)\,ds\,dx\,dv + \int_0^t\int_{\Lambda\times\mathbf{R}^3}[\partial_s f^{\#}(s)]\log f^{\#}(s)\,ds\,dx\,dv \tag{3.9}$$

Here by Eq. (3.7), we have

$$\int_0^t \int_{\Lambda \times \mathbf{R}^3} \partial_s f^{\#}(s)\, ds\, dx\, dv$$

$$= \int_{\Lambda \times \mathbf{R}^3} f(t)\, dx\, dv - \int_{\Lambda \times \mathbf{R}^3} f_0\, dx\, dv = -\varepsilon \int_0^t \int_M ff_+ B\, ds\, d\mu \leqslant 0 \qquad (3.10)$$

and so

$$\int_{\Lambda \times \mathbf{R}^3} f(t) \log f(t)\, dx\, dv \leqslant \int_{\Lambda \times \mathbf{R}^3} f_0 \log f_0\, dx\, dv$$
$$+ \int_0^t \int_{\Lambda \times \mathbf{R}^3} [Q^n(f)]^{\#}\,(s) \log f^{\#}(s)\, ds\, dx\, dv \qquad (3.11)$$

But

$$\int_0^t \int_{\Lambda \times \mathbf{R}^3} [Q^n(f)]^{\#}\,(s) \log f^{\#}(s)\, ds\, dx\, dv$$
$$\leqslant \frac{\varepsilon}{2} \int_0^t \int_M [ff_+ \log^-(ff_+)]^{\#}\,(s)\, B\, ds\, d\mu$$
$$+ \frac{\kappa}{2} \int_0^t \int_M (ff_- - ff_+)^{\#}\,(s)\, B\, ds\, d\mu$$
$$+ \frac{\kappa}{2} \int_0^t \int_M (1-\chi_n)(ff_+)^{\#}\,(s)\, B\, ds\, d\mu$$
$$\equiv I_1 + I_2 + I_3 \qquad (3.12)$$

Now Eq. (3.8) and the elementary inequality

$$-y \log y \leqslant \psi y + \exp(-\psi - 1) \qquad \text{if} \quad y, \psi > 0$$

imply that

$$I_1 \leqslant \frac{\varepsilon}{2} \int_0^t \int_M [ff_+(v^2 + v_*^2) + \exp(-v^2 - v_*^2 - 1)]\, B\, ds\, d\mu$$
$$\leqslant \int_0^t \int_{\Lambda \times \mathbf{R}^3} f_0 v^2\, dx\, dv + \frac{\pi \varepsilon t \sigma^2}{3e} \left\{ \int_{\mathbf{R}^3} [(|v| + 1) \exp(-v^2)]\, dv \right\} \qquad (3.13)$$

Moreover, $I_2$ can be bounded as in (2.8):

$$I_2 \leqslant \frac{\kappa\sigma^2}{2}\left(\int_{\Lambda \times \mathbf{R}^3} f_0 \, dx \, dv\right)^2 \tag{3.14}$$

and an estimate for $I_3$ follows from Eq. (3.7):

$$I_3 \leqslant \frac{\kappa}{2\varepsilon}\int_{\Lambda \times \mathbf{R}^3} f_0 \, dx \, dv \tag{3.15}$$

Finally, Eqs. (3.11)–(3.15) imply the following entropy estimate:

$$\begin{aligned}
&\int_{\Lambda \times \mathbf{R}^3} f(t) \log f(t) \, dx \, dv \\
&\quad \leqslant \int_{\Lambda \times \mathbf{R}^3} f_0 \left(\log f_0 + v^2 + \frac{\kappa}{2\varepsilon}\right) dx \, dv \\
&\qquad + \frac{\kappa\sigma^2}{2}\left(\int_{\Lambda \times \mathbf{R}^3} f_0 \, dx \, dv\right)^2 + \frac{\pi\varepsilon t\sigma^2}{3e}\left\{\int_{\mathbf{R}^3} [(|v|+1)\exp(-v^2)] \, dv\right\}^2
\end{aligned} \tag{3.16}$$

to be used later.

In this paper we shall use various forms of solution: the renormalized, mild, and exponential multiplier forms as defined in, e.g., ref. 1, together with the following iterated integral form introduced in ref. 4:

**Definition.** $f$ satisfies the EE in *iterated integral form* if

$$Q^{\pm}(f)^{\#}(x, v, \cdot) \in L^1_{\mathrm{loc}}(\mathbf{R}_+) \quad \text{for a.a.} \quad (x, v) \in \Lambda \times \mathbf{R}^3 \tag{3.17}$$

and

$$\begin{aligned}
&\int_{\Lambda \times \mathbf{R}^3} f^{\#}(t) \, \psi(t) \, dx \, dv \\
&\quad = \int_{\Lambda \times \mathbf{R}^3} f_0 \psi(0) \, dx \, dv + \int_0^t \int_{\Lambda \times \mathbf{R}^3} f^{\#}(s) \, \partial_s \psi \, dx \, dv \, ds \\
&\qquad + \int_{\Lambda \times \mathbf{R}^3} \left[\int_0^t \psi(s) \, Q(f)^{\#} \, (s) \, ds\right] \mathrm{x} \, \mathrm{dv}, \qquad t > 0, \quad \psi \in CL
\end{aligned} \tag{3.18}$$

Here $CL$ is the linear space of all functions $\psi$ in $C^1(\mathbf{R}_+; L^\infty(\Lambda \times \mathbf{R}^3))$ with bounded support and with $\psi(x, v, \cdot) \in C^1(R_+)$ for a.a. $(x, v) \in \Lambda \times \mathbf{R}^3$. The last integral is an iterated integral. It is not required that

$|\psi Q(f)^{\#}| \in L^1(\Lambda \times \mathbf{R}^3 \times (0, t))$, only that $\int_0^t \psi(s)\, Q(f)^{\#}\,(s)\, ds \in L^1(\Lambda \times \mathbf{R}^3)$. Finally,

$$Q^+(f) = \int_{\mathscr{S}_+ \times \mathbf{R}^3} f'f'_- \kappa B(v, v_*, u)\, dv_*\, du \tag{3.19}$$

$$Q^-(f) = \int_{\mathscr{S}_+ \times \mathbf{R}^3} ff_+ \kappa B(v, v_*, u)\, dv_*\, du \tag{3.20}$$

**Lemma 1.** The above four solution forms are equivalent for the EE if

$$\frac{Q^{\pm}(f)}{1+f}, \int_{\mathscr{S}_+ \times \mathbf{R}^3} f_+ B\, dv_*\, du \in L^1_{\mathrm{loc}}(\Lambda \times \mathbf{R}^3 \times \mathbf{R}_+) \tag{3.21}$$

The proof of the lemma was given in ref. 7 for the symmetrized case, but goes through unmodified in the unsymmetrized case. ∎

## 4. A PRELIMINARY STUDY OF EQUATION (3.4)

The aim of this section is to study the well-posedness of the initial value problem for Eq. (3.4) in $\Lambda \times \mathbf{R}^3$ by means of estimates of the type introduced in ref. 6. We start with some local results.

**Lemma 2.** Suppose $f_0 \in L^1_+(\Lambda \times \mathbf{R}^3)$, $f_0(x, v) = 0$ for $|v| \geqslant 2n$. Then there is a unique solution of Eq. (3.4) with initial value $f_0$ and $\sup_{0 \leqslant t \leqslant T'} f^{\#}(x, v, t) \in L^1_+(\Lambda \times \mathbf{R}^3)$ for $T'$ small enough. This solution satisfies Eqs. (3.7) and (3.8).

*Proof.* Set ${}_0f = 0$, and define inductively for $j \in \mathbf{N}$

$$L_j f(t) = \int_0^t \int_{\mathscr{S}_+ \times \mathbf{R}^3} {}_jf(x + \sigma u + vs, v_*, s)(\varepsilon + \kappa\chi_n)\, B(v, v_*, u)\, dv_*\, du \tag{4.1}$$

$$\begin{aligned} {}_{j+1}f(t) &= f_0 \exp[-L_j f(t)] \\ &\quad + \int_0^t \exp[-L_j f(t) + L_j f(s)] \int_{\mathscr{S}_+ \times \mathbf{R}^3} \kappa({}_jf'\,{}_jf'_-)^{\#}\, \chi_n B\, ds\, dv_*\, du \end{aligned} \tag{4.2}$$

It follows that for $j \in \mathbf{N}$

$${}_jf \geqslant 0, \qquad {}_jf(x, v, t) = 0 \qquad \text{for} \quad |v| \geqslant 2n \tag{4.3}$$

and similarly to Section 3

$$
{}_{j+1}f^{\#}(t) = f_0 + \int_0^t \int_{\mathscr{S}_+ \times \mathbf{R}^3} \{\kappa\chi_n(v, v_*)\, {}_jf'\, {}_jf'_- - [\kappa\chi_n(v, v_*) + \varepsilon]\, {}_{j+1}f_j f_+\}\, B(v, v_*, u)\, ds\, dv_*\, du \tag{4.4}
$$

Set

$$
f_i^{\#}(x, v, t) = f_{i0}(x, v), \qquad f_e = f - f_i, \qquad f_{e0} = f_0 - f_{i0} \tag{4.5}
$$

where

$$
f_{i0}(x, v) = \chi(f_0(x, v)/\omega)\, f_0(x, v) \tag{4.6}
$$

Choose $\omega > n$ so large that

$$
\|f_{e0}\| = \int_{\Lambda \times \mathbf{R}^3} f_{e0}\, dx\, dv < 128^{-1} \tag{4.7}
$$

and set

$$
\|f\|_{T'} = \int_{\Lambda \times \mathbf{R}^3} \sup_{0 \leqslant t \leqslant T'} |f^{\#}(x, v, t)|\, dx\, dv \tag{4.8}
$$

If we split ${}_jf$ as $= f_i + {}_jf_e$, straightforward estimates of each term separately in Eq. (4.4) give

$$
\begin{aligned}
\|{}_{j+1}f_e\|_{T'} \leqslant{}& \|f_{e0}\| + 3\mathscr{S}(2n\pi\sigma^2 T', f_{i0})\, \|f_{i0}\| \\
&+ \|{}_jf_e\|_{T'}\, [\|{}_jf_e\|_{T'} + 2\, \|{}_{j+1}f_e\|_{T'} + 4\mathscr{S}(2n\pi\sigma^2 T', f_{i0})] \\
&+ 2\mathscr{S}(2n\pi\sigma^2 T', f_{i0})\, \|{}_{j+1}f_e\|_{T'}
\end{aligned} \tag{4.9}
$$

Here

$$
\mathscr{S}(\delta, f) = \sup_{M(\delta)} \int_M |f(x, v)|\, dx\, dv \tag{4.10}
$$

where $M(\delta)$ is the set of all measurable subsets $M \subset \Lambda \times \mathbf{R}^3$ such that for a.e. $v \in \mathbf{R}^3$, the set $M_v$ of those $x$ for which $(x, v) \in M$ has measure less than $\delta$. The details of estimates of the above type can be found in ref. 6. We remark that the use of the norm (4.8) is based on the fact that for fixed $x$, $v$, and $v_*$ the Jacobian of the transformation

$$
(u, s) \to y = x + s(v - v_*) \pm \sigma u
$$

is $\pm[\sigma^2(u, v-v_*)]^{-1}$. This was noticed by Cercignani[5] and used by Arkeryd.[6] In the case of a bounded domain $\Lambda$ this transformation is not one-to-one, in general; if, however, we restrict ourselves to sufficiently small values of $T'$ as assumed, the transformation is actually one-to-one, because if $4nT'<1$, then $|v-v_*|\,T'<1$.

We now choose $T'$ so that $2nT'<1$, $\mathscr{S}(2n\pi\sigma^2T', \mathscr{B})<16^{-1}$, $3\mathscr{S}(2n\pi\sigma^2T', \mathscr{B})\,\|\mathscr{B}\|<128^{-1}$, with $\mathscr{B}(x, v)=2\omega$ for $x\in\Lambda$, $|v|\leqslant 2n$, and $\mathscr{B}(x, v)=0$ otherwise. It follows that $\|_jf_e\|_{T'}<16^{-1}$ for $j\in\mathbf{N}$. Moreover, for the same value of $T'$,

$$\begin{aligned}
&\|_{j+1}f_e-{}_{m+1}f_e\|_{T'}\\
&\quad\leqslant 4\mathscr{S}(2n\pi\sigma^2T', f_{i0})\,\|_jf_e-{}_mf_e\|_{T'}\\
&\qquad+2\mathscr{S}(2n\pi\sigma^2T', f_{i0})\,\|_{j+1}f_e-{}_{m+1}f_e\|_{T'}+(\|_jf_e\|_{T'}+\|_mf_e\|_{T'}\\
&\qquad+2\,\|_{m+1}f_e\|_{T'})\,\|_jf_e-{}_mf_e\|_{T'}+2\,\|_jf_e\|_{T'}\,\|_{j+1}f_e-{}_{m+1}f_e\|_{T'}\\
&\quad\leqslant \tfrac{1}{4}\,\|_{j+1}f_e-{}_{m+1}f_e\|_{T'}+\tfrac{1}{2}\,\|_jf_e-{}_mf_e\|_{T'}
\end{aligned}\tag{4.11}$$

Hence $({}_jf_e)_{j\in N}$ is Cauchy in the $\|\cdot\|_{T'}$-norm. Denote the limit by $f_e^n$. It follows that Eq. (3.4) has a unique nonnegative solution $f^n=f_i+f_e^n$ on $[0, T']$ with $\|f^n\|_{T'}<\infty$. Each of the time-integrated gain and loss terms belongs to $L^1_+$, and so the changes of variables leading to (3.7) and (3.8) hold in a strict sense; thus, $f_n$ satisfies Eqs. (3.7) and (3.8). ∎

**Lemma 3.** Let $f$ be the solution of Lemma 2 with initial value $f_0$. If $\partial^\alpha f_0\in L^1(\Lambda\times\mathbf{R}^3)$, $|\alpha|\leqslant k$, then $\partial^\alpha f\in L^1(\Lambda\times\mathbf{R}^3)$, $|\alpha|\leqslant k$, $t<T'$, and $\|\partial^\alpha f\|_{T'}<\infty$.

*Proof.* (i) $\partial^\alpha=\partial_{x_1}, \partial_{x_2}$, or $\partial_{x_3}$. Consider the equation

$$\begin{aligned}
g^\#(t)=\partial^\alpha f_0+\int_0^t\int_{\mathscr{S}_+\times\mathbf{R}^3}&\{\kappa\chi_n(v, v_*)(g'f'_-+f'g'_-)^\#\,(s)\\
&-[\kappa\chi_n(v, v_*)+\varepsilon](gf_++fg_+)^\#\}\,B(v, v_*, u)\,ds\,dv_*\,du
\end{aligned}\tag{4.12}$$

Contraction mapping estimates of the type in Lemma 2 prove the existence of a unique solution of Eq. (4.12) on $[0, T']$ with $\|g\|_{T'}<\infty$. The fact that $g=0$ for $|v|\geqslant 2n$ follows from estimates of the type used in Lemma 2, in particular the estimate (4.3), which follows from the exponential form (4.2). The difference quotient $\Delta^\alpha f/\Delta x^\alpha$ solves a related equation and converges by the same type of estimates to $g$ in the $\|\cdot\|_{T'}$-norm, when $\Delta x\to 0$. Hence $\partial^\alpha f\in C_0([0, T'], L^1)$.

(ii) $\partial^\alpha=\partial_{v_1}, \partial_{v_2}$, or $\partial_{v_3}$. Besides terms as in (i) related to a factor

"$\partial^\alpha f$" in the $Q$-integral, this case involves terms originating from "$\partial^\alpha B$, $\partial^\alpha \chi_n B$." We have to control

$$\mathscr{T} = \int_0^t \int_{\mathscr{S}_+ \times \mathbf{R}^3} [\kappa \, |\partial^\alpha(\chi_n B)| \, (f' f'_-)^{\#} \, (s)$$

$$+ (\kappa \, |\partial^\alpha \chi_n B| + \varepsilon \, |\partial^\alpha B|)(f f_+)^{\#} \, (s)] \, ds \, dv_* \, du \tag{4.13}$$

and the corresponding difference quotients. These terms cannot be bounded by the estimates of Lemma 2, since the required $B$ factor is now absent. Instead we use the control of $\partial_{x_1}$ from $i$, together with the estimate (Sobolev imbedding)

$$\int_{S \times \mathbf{R}^3} \sigma^2 f(x + \sigma u, v_*, t) \, dv_* \, du$$

$$\leqslant C \int_{\Lambda \times \mathbf{R}^3} \left[ |f(x, v_*, t)| + \sum_{i=1}^3 |\partial_{x_i} f(x, v_*, t)| \right] dv_* \, dx$$

$$\leqslant C \left( \|f\|_{T'} + \sum_{i=1}^3 \|\partial_{x_i} f\|_{T'} \right) \tag{4.14}$$

It follows that

$$\int_{\Lambda \times \mathbf{R}^3} \mathscr{T} \, dx \, dv \leqslant CT' \, \|f\|_{T'} \left( \|f\|_{T'} + \sum_{i=1}^3 \|\partial_{x_i} f\|_{T'} \right) \tag{4.15}$$

Using (4.15) in the relevant places, the proof of (i) can now be repeated to prove that $\partial^\alpha f \in C_0([0, T'], L^1)$.

(iii) Higher-order derivatives. The arguments of (i) and (ii) can now be repeated to prove the lemma for $|\alpha| = 2$, and then successively in the same way for higher-order derivatives. ∎

**Lemma 4.** Let $f$ be the solution of Lemma 2 with initial value $f_0$. If $f_0 \log f_0 \in L^1(\Lambda \times \mathbf{R}^3)$, then $f(t)$ satisfies Eq. (3.16) for $t \leqslant T'$.

*Proof.* If $f_0$ is smooth enough, then it follows from Lemma 3 that the solution $f$ is smooth enough on $[0, T'] \times \Lambda \times \mathbf{R}^3$ for the formal computations of (3.11)–(3.15) to hold in a strict sense, thus implying Eq. (3.16).

Consider next an arbitrary $f_0$ of the type considered in the present lemma. Introduce a sequence of smooth initial values $f_{0\nu} \to f_0$ in $L^1_+$ when $\nu \to \infty$, and with $f_{0\nu} = 0$ for $|v| \geqslant 2n$. By the proof of Lemma 2 the corresponding solutions $f_\nu$ are for $\nu$ large enough defined in $[0, T']$, and $\lim_{\nu \to \infty} \|f - f_\nu\|_{T'} = 0$. In particular, we can take $f_{0\nu}$ with

$$\int_{\Lambda \times \mathbf{R}^3} f_{0\nu} \log f_{0\nu} \, dx \, dv \to \int_{\Lambda \times \mathbf{R}^3} f_0 \log f_0 \, dx \, dv \tag{4.16}$$

Then (3.16) follows for $f$ on $[0, T']$. ∎

The previous lemmas imply global well-posedness for Eq. (3.4) under (2.5).

**Theorem 5.** Suppose $f_0 \in L^1_+(\Lambda \times \mathbf{R}^3)$ with $f_0 \log f_0 \in L^1_+(\Lambda \times \mathbf{R}^3)$, and $f_0 = 0$ for $|v| \geqslant 2n$. Then there is a unique solution of Eq. (3.4) for $t > 0$ with initial value $f_0$. For $t > 0$ the solution satisfies Eqs. (3.7), (3.8), and (3.16) as well as the entropy estimate

$$\begin{aligned}
&\int_{\Lambda \times \mathbf{R}^3} f(t) \log f(t) \, dx \, dv \\
&\quad \leqslant \int_{\Lambda \times \mathbf{R}^3} f_0(\log f_0 + v^2) + dx \, dv \\
&\quad\quad + \frac{\kappa \sigma^2}{2} \left( \int_{\Lambda \times \mathbf{R}^3} f_0 \, dx \, dv \right)^2 + \frac{\pi \varepsilon t \sigma^2}{3e} \left\{ \int_{\mathbf{R}^3} (|v| + 1) \exp(-v^2)] \, dv \right\}^2 \\
&\quad\quad + \frac{\kappa}{2} \int_0^t \int_M (1 - \chi_n)(ff_+)^{\#} \, (s) \, B \, ds \, d\mu
\end{aligned} \tag{4.17}$$

*Proof.* We observe that by Lemma 4 for $0 \leqslant t \leqslant T'$

$$\begin{aligned}
&\int_{f > \omega} f(t) \, dx \, dv \\
&\quad \leqslant \frac{1}{\log \omega} \int_{\Lambda \times \mathbf{R}^3} f \log^+ f \, dx \, dv \\
&\quad \leqslant \frac{1}{\log \omega} \left( \int_{\Lambda \times \mathbf{R}^3} f \log f \, dx \, dv + \frac{32 \pi n^3}{3e} \right) \\
&\quad \leqslant \frac{1}{\log \omega} \left\{ \int_{\Lambda \times \mathbf{R}^3} f_0 \left( \log f_0 + v^2 + \frac{\kappa}{2\varepsilon} \right) dx \, dv + \frac{32 \pi n^3}{3e} \right. \\
&\quad\quad \left. + \frac{\kappa \sigma^2}{2} \left( \int_{\Lambda \times \mathbf{R}^3} f_0 \, dx \, dv \right)^2 + \frac{\pi \varepsilon t \sigma^2}{3e} \left[ \int_{\mathbf{R}^3} (|v| + 1) \exp(-v^2) \, dv \right]^2 \right\}
\end{aligned} \tag{4.18}$$

Given any time interval $[0, T]$, $\omega$ can be chosen in such a way that the right-hand side of Eq. (4.18) is less than $128^{-1}$ for $t$ in $[0, T]$. The time interval $[0, T']$ of Lemma 2 can be chosen with respect to this $\omega$, and the solution $f$ will satisfy Eqs. (3.7), (3.8), and (3.16) on $[0, T']$. Lemma 2 can next be applied to $[T', 2T']$ with initial value $f(x, v, T')$ and then successively on subintervals of length $T'$ covering $[0, T]$.

Moreover, the argument in the proof of Lemma 4 can be applied while keeping the term

$$\frac{\kappa}{2}\int_0^t \int_M (1-\chi_n)(ff_+)^{\#}(s)\, B\, ds\, d\mu$$

throughout the limits, thus proving Eq. (4.17) on $[0, T]$. This completes the proof of the theorem; in fact the solution is unique and the result holds for $t > 0$. ∎

**Theorem 6.** The solution $f$ of Theorem 5 also satisfies

$$\begin{aligned}\int_{O\times\mathscr{S}_+} &\frac{\kappa}{2} f'f'_-\chi_n B\, ds\, d\mu \\ &\leqslant j\int_{O\times\mathscr{S}_+} \frac{\kappa}{2} ff_-\chi_n B\, ds\, d\mu \\ &\quad + \frac{1}{l(j)}\left\{\int_{\Lambda\times\mathbf{R}^3} f_0(\log^+ f_0 + 2v^2)\, dx\, dv + \frac{\kappa\sigma^2}{2}\left(\int_{\Lambda\times\mathbf{R}^3} f_0\, dx\, dv\right)^2\right. \\ &\quad \left. + \int_{\mathbf{R}^3}\exp(-v^2-1)\, dv + \frac{\pi\varepsilon t\sigma^2}{3e}\left[\int_{\mathbf{R}^3}(|v|+1)\exp(-v^2)\, dv\right]^2\right\} \end{aligned} \tag{4.19}$$

Here $j > 1$, and $O$ is a measurable set in $\Lambda\times\mathbf{R}^3\times\mathbf{R}^3\times[0, T]$.

*Proof.* Evidently

$$\begin{aligned}\int_{O\times\mathscr{S}_+} &\frac{\kappa}{2} f'f'_-\chi_n B\, ds\, d\mu \\ &\leqslant j\int_{O\times\mathscr{S}_+} \frac{\kappa}{2} ff_-\chi_n B\, ds\, d\mu \\ &\quad + \frac{1}{l(j)}\int_0^T\int_M \frac{\kappa}{2} f'f'_-\, l\left(\frac{f'f'_-}{ff_-}\right)\chi_n B\, ds\, d\mu \end{aligned} \tag{4.20}$$

By (2.9) and (3.13), in the case of a smooth $f_0$

$$
\begin{aligned}
&\int_0^T \int_M \frac{\kappa}{2} f'f'_- l\left(\frac{f'f'_-}{ff_-}\right) \chi_n B\, ds\, d\mu \\
&\quad = -\int_0^T \frac{dH}{ds}(s)\, ds - \frac{\varepsilon}{2}\int_M ff_+ \log ff_+ B\, ds\, d\mu \\
&\quad \leqslant \int_{\Lambda\times\mathbf{R}^3} f_0(\log f_0 + v^2)\, dx\, dv + \frac{\kappa\sigma^2}{2}\left(\int_{\Lambda\times\mathbf{R}^3} f_0\, dx\, dv\right)^2 \\
&\qquad + \int_{\mathbf{R}^3} f(T)\log^- f(T)\, dx\, dv + \frac{\pi\varepsilon t\sigma^2}{3e}\left[\int_{\mathbf{R}^3} (|v|+1)\exp(-v^2)\, dv\right]^2 \\
&\quad \leqslant \int_{\Lambda\times\mathbf{R}^3} f_0(\log f_0 + 2v^2)\, dx\, dv + \frac{\kappa\sigma^2}{2}\left(\int_{\Lambda\times\mathbf{R}^3} f_0\, dx\, dv\right)^2 \\
&\qquad + \int_{\mathbf{R}^3} \exp(-v^2-1)\, dv + \frac{\pi\varepsilon t\sigma^2}{3e}\left[\int_{\mathbf{R}^3} (|v|+1)\exp(-v^2)\, dv\right]^2 \qquad (4.21)
\end{aligned}
$$

In the case of smooth data $f_0$, Eq. (4.19) follows from Eqs. (4.20) and (4.21). Using the continuous $L^1$ dependence of the solution on the initial data, together with Fatou's lemma, Eq. (4.19) follows for an arbitrary $f_0 \in L^1_+(\Lambda\times\mathbf{R}^3)$ with $f_0 \log f_0 \in L^1_+(\Lambda\times\mathbf{R}^3)$ and $f_0 = 0$ for $|v| \geqslant 2n$. ∎

*Remark.* From Eq. (3.4) in the equivalent exponential form it immediately follows that for a.e. $(x, v) \in \Lambda\times\mathbf{R}^3$

$$
f(x, v, T)\exp\left(\int_0^T h_n^{\#}(x, v, s)\, ds\right) \geqslant f^{\#}(x, v, t), \qquad 0 \leqslant t \leqslant T \qquad (4.22)
$$

Here

$$
h_n^{\#}(x, v, t) = \int_{\mathscr{S}_+\times\mathbf{R}^3} f(x + \sigma u + vs, v_*, s)(\varepsilon + \kappa\chi_n)\, B(v, v_*, u)\, dv_*\, du \qquad (4.23)
$$

## 5. AN EXISTENCE THEOREM

Having established the relevant preliminary results, we can now proceed to a discussion of the proof of an existence theorem for the EE (2.1) by the type of argument given in ref. 7 for the symmetrized Enskog equation. The actual proof becomes more technical in the present case due to two consecutive approximations with different and more involved entropy estimates.

Given $f_0$ satisfying Eq. (2.5), we obtain the approximating sequence (3.4) with initial value $f_{0n} = \chi(|v|/n) f_0$. By Theorem 5 the corresponding solutions exists and satisfies Eqs. (3.7), (3.8), and (3.16) Applying these estimates together with (4.22) and (4.19) to the mild form of (3.4) gives that $(f^n)_{n \in \mathbf{N}}$ is uniformly equicontinuous from $[0, T]$ to $L^1(\Lambda \times \mathbf{R}^3)$. Hence there is a subsequence $(f^{n'})$ converging weakly in $L^1(\Lambda \times R^3 \times [0, T])$ as well as in $L^1(\Lambda \times \mathbf{R}^3)$ for $0 \leqslant t \leqslant T$ to a function $f \in C([0, T], L^1_+(\Lambda \times \mathbf{R}^3))$. Vavious subsequences of $(f^{n'})$ will also be denoted by $(f^{n'})$ in the sequel without any further comment.

The averaging technique of Golse *et al.*[13,14] will be used in the form stated in ref. 1:

**Lemma 7.**[13,14,1] Let $(E, \mu)$ be an arbitrary measure space, and let $\psi \in L^\infty(\Lambda \times \mathbf{R}^3 \times [0, T]; L^1(E))$.

(i) If $g^n$ and $G^n$ belong to a weakly compact set in $L^1(K)$ for any compact set $K$ in $\Lambda \times \mathbf{R}^3 \times (0, T)$, and $(\partial_t + v\partial_x) g^n = G^n$ in distribution sense, $n \in \mathbf{N}$, then $\int_{\mathbf{R}^3} g^n \psi \, dv$ belongs to a compact set in $L^1(\Lambda \times (0, T) \times E)$ for any compact set $K$ in $\Lambda \times \mathbf{R}^3 \times (0, T)$, provided that supp $\psi \subset K \times E$.

(ii) If in addition $g^n$ belongs to a weakly compact set in $L^1(\Lambda \times \mathbf{R}^3 \times [0, T])$, $n \in \mathbf{N}$, then $\int_{\mathbf{R}^3} g^n \psi \, dv$ belongs to a compact set in $L^1(\Lambda \times [0, T] \times E)$.

We shall apply this lemma to the renormalized, approximated Enskog equation

$$(\partial_t + v\partial_x) g_\delta = Q^n_\delta(g) \tag{5.1}$$

with

$$g_\delta = \delta^{-1} \log(1 + \delta g), \qquad Q^n_\delta(g) = Q^n(g)/(1 + \delta g) \tag{5.2}$$

Equation (5.1) is satisfied in the distribution sense by $g = f^n$ and, in order to apply the lemma, $f^n_\delta$ and $Q^n_\delta$ should satisfy the conditions on $g^n$ and $G^n$ in Lemma 7. But $f^n_\delta$, $n \in \mathbf{N}$, belongs to a weakly compact set in $L^1(\Lambda \times \mathbf{R}^3 \times [0, T])$ since $0 \leqslant f^n_\delta \leqslant f^n$, and, as discussed above, $\{f^n\}_{n \in \mathbf{N}}$ is weakly precompact in $L^1$. Evidently, by (3.16)

$$\{f^n f^n_+ \chi(x, v) \, |v - v_*|/(1 + \delta f^n)\}_{n \in \mathbf{N}}$$

is weakly precompact in $L^1(M \times [0, T])$ for any characteristic function $\chi$ of a measurable set of bounded support in $\Lambda \times \mathbf{R}^3$.

It follows that for $\delta > 0$, $\{Q^{n-}_\delta(f^n)\}_{n \in \mathbf{N}}$ is a weakly precompact subset of $L^1(K)$ for any compact subset $K$ of $\Lambda \times \mathbf{R}^3 \times [0, T]$. This, together with (4.19), implies the same weak $L^1$-precompactness for $\{Q^{n+}_\delta(f^n)\}_{n \in \mathbf{N}}$, when

$\delta > 0$. So we have the relevant compactness properties of $\{f_\delta^n\}_{n\in\mathbf{N}}$ and $\{Q_\delta^n(f^n)\}_{n\in\mathbf{N}}$ for an application of Lemma 7.

Hence, for any $\psi \in L^\infty(\Lambda \times \mathbf{R}^3 \times [0, T])$, $\delta > 0$,

$$\lim_{n'} \int_{\mathbf{R}^3} f_\delta^{n'} \psi \, dv = \int_{\mathbf{R}^3} f_\delta \psi \, dv \qquad \text{in} \quad L^1(\Lambda \times [0, T]) \tag{5.3}$$

Here $f_\delta$ is the weak $L^1$-limit of $\{f_\delta^{n'}\}$. It follows (by means of an argument first used in ref. 4 and, more recently, in ref. 1) from (3.16) that

$$\lim_{\delta \to 0+} \sup_n \int_0^T ds \int_{\Lambda \times \mathbf{R}^3} |f_\delta^n - f^n| \, dx \, dv = 0 \tag{5.4}$$

and that

$$\lim_{\delta \to 0+} \int_0^T ds \int_{\Lambda \times \mathbf{R}^3} |f_\delta - f| \, dx \, dv = 0 \tag{5.5}$$

This together with Eq. (5.3) gives

$$\lim_{n'} \int_{\mathbf{R}^3} f^{n'} \psi \, dv = \int_{\mathbf{R}^3} f \psi \, dv \qquad \text{in} \quad L^1(\Lambda \times [0, T]) \tag{5.6}$$

Thus, a change of variables $x \to x \pm \sigma u$ and another application of the averaging argument implies the following result.

**Lemma 8.** We have

$$\lim_{n'} \int_{\mathbf{R}^3} f^{n'}(x \pm \sigma u, v_*, t) \, B\psi \, dv_*$$

$$= \int_{\mathbf{R}^3} f(x \pm \sigma u, v_*, t) \, B\psi \, dv_*$$

$$\text{in} \quad L^1_{\text{loc}}(\Lambda \times \mathbf{R}^3 \times \mathscr{S}_+ \times [0, T]) \qquad \text{for} \quad \psi \in L^\infty(M \times [0, T]) \tag{5.7}$$

Similarly to Lemma 1, the mild solution $f^n$, $n \in \mathbf{N}$, of the truncated EE also satisfies a renormalized iterated integral form of the equation:

$$\int_{\Lambda \times \mathbf{R}^3} f_\delta^{n\#}(t) \, \psi(t) \, dx \, dv$$

$$= \int_{\Lambda \times \mathbf{R}^3} f_{0\delta}^n \psi(0) \, dx \, dv + \int_0^t \int_{\Lambda \times \mathbf{R}^3} f_\delta^{n\#}(s) \, \partial_s \psi \, dx \, dv \, ds$$

$$+ \int_{\Lambda \times \mathbf{R}^3} \left[ \int_0^t \psi(s) \, Q_\delta^n(f)^\# (s) \, ds \right] dx \, dv, \qquad t > 0, \quad \psi \in CL \tag{5.8}$$

Consider the limit of each term in Eq. (5.8) for $n' \to \infty$ and then as $\delta \to 0+$. Arguing in the same way as we did before to prove (5.6), the first three terms of Eq. (5.8) then converge to

$$\int_{\Lambda \times \mathbf{R}^3} f^{\#}(t)\, \psi(t)\, dx\, dv, \quad \int_{\Lambda \times \mathbf{R}^3} f_0 \psi(0)\, dx\, dv, \quad \int_0^t \int_{\Lambda \times \mathbf{R}^3} f^{\#}(s)\, \partial_s \psi\, dx\, dv\, ds$$

Notice that (4.22) and (5.7) imply that

$$\begin{aligned} f^{\#}(x, v, T) \exp\left[\int_0^T h^{\#}(x, v, \tau)\, d\tau\right] \\ \geqslant f^{\#}(x, v, t) \quad \text{on} \quad 0 \leqslant t \leqslant T \quad \text{a.e. in } (x, v) \end{aligned} \tag{5.9}$$

Here

$$h^{\#}(x, v, t) = \int_{\mathscr{S}_+ \times \mathbf{R}^3} f(x + \sigma u + vs, v_*, s)(\varepsilon + \kappa)\, B(v, v_*, u)\, dv_*\, du \tag{5.10}$$

Then a proof of the following existence result for the initial value problem of the EE can be reduced to a study of the limit of the collision term in Eq. (5.8).

**Theorem 9.** Let $f_0$ satisfy (2.5). Then there exists a function $f \in C((0, \infty); L^1(\Lambda \times \mathbf{R}^3))$ satisfying the equation

$$(\partial_t + v\partial_x) f + \varepsilon \int_{\mathscr{S}_+ \times \mathbf{R}^3} ff_+ B(v, v_*, u)\, dv_*\, du = Q(f) \tag{5.11}$$

(in any of the four forms of Lemma 1) with initial value $f_0$, such that (3.16) holds for $t \geqslant 0$. For $t > 0$ mass and energy are bounded by their initial values, and the entropy satisfies

$$\begin{aligned} &\int_{\Lambda \times \mathbf{R}^3} f(t) \log f(t)\, dx\, dv \\ &\quad \leqslant \int_{\Lambda \times \mathbf{R}^3} f_0(\log f_0 + v^2)\, dx\, dv \\ &\qquad + \frac{\kappa\sigma^2}{2}\left(\int_{\Lambda \times \mathbf{R}^3} f_0\, dx\, dv\right)^2 + \frac{\pi\varepsilon t\sigma^2}{3e}\left[\int_{\mathbf{R}^3} (|v| + 1) \exp(-v^2)\, dv\right]^2 \end{aligned} \tag{5.12}$$

Finally, (5.9) holds for $T > 0$.

*Proof.* Starting with the loss term, the loss and gain parts of the collision term will be studied separately. For the loss term, given $\varepsilon > 0$, the integral $\int_0^T h^{\#}(x, v, \tau)\, d\tau$ and $f^{\#}(x, v, t)$ are locally bounded on the complement $\Omega \subset \Lambda \times \mathbf{R}^3$ of some set of measure $\leqslant \varepsilon$. Then, for $\psi \in CL$ with $\psi = 0$ outside of $\Omega \times \mathbf{R}_+$:

$$\begin{aligned}
\lim \int_0^t ds \int \left(\frac{f^{n'} f_+^{n'}}{1+\delta f^{n'}}\right)^{\#} \psi(\varepsilon + \kappa\chi_n)\, B\, d\mu \\
= \lim \int_0^t ds \int \left(\frac{f^{n'} f_+^{n'}}{1+\delta f^{n'}}\right)^{\#} \psi(\varepsilon + \kappa)\, B\, d\mu \\
= \int ds \int (ff_+)^{\#}\, \psi(\varepsilon + \kappa)\, B\, d\mu, \qquad t \leqslant T
\end{aligned} \tag{5.13}$$

Actually, it would have been enough to consider $\psi \geqslant 0$ in the above limit of the loss term of Eq. (5.8). That is also the case for the gain term. Accordingly, the proof is complete for $0 \leqslant t \leqslant T$ and $\psi$ with support in $\Omega \times \mathbf{R}_+$, once the following two lemmas are proved for such $\psi$, $\psi \geqslant \geqslant 0$.

**Lemma 10.** The following relation holds:

$$\lim \int_0^t ds \int_M \left(\frac{(f^{n'} f_-^{n'})}{1+\delta f^{n'}}\right)^{\#} \psi B \chi_n\, d\mu \geqslant \int_0^t ds \int_M (f' f'_-)^{\#}\, \psi B\, d\mu, \qquad t \leqslant T \tag{5.14}$$

**Lemma 11.** The following relation holds:

$$\lim \int_0^t ds \int_M \left(\frac{(f^{n'} f_-^{n'})'}{1+\delta f^{n'}}\right)^{\#} \psi B \chi_n\, d\mu \leqslant \int_0^t ds \int_M (f' f'_-)^{\#}\, \psi B\, d\mu, \qquad t \leqslant T \tag{5.15}$$

The limit in the lemmas exist, since we have already proved that the limits of each of the other terms in Eq. (5.8) are finite. Also, from Lemma 5 (which will be proved first) it follows that the right-hand side in Lemma 6 is finite.

Once the lemmas are proved, and thus $f$ satisfies Eq. (3.18), with $\varepsilon + \kappa$ in place of $\kappa$ in the loss term, for $t \leqslant T$ and $\psi$ with support in $\Omega \times \mathbf{R}_+$, then Eq. (3.18) (with $\varepsilon + \kappa$ in place of $\kappa$ in the loss term) follows for any $\psi \in CL$ and $t \leqslant T$ by approximation.

Indeed, the integral $I(x, v) = \int_0^t \psi Q(f, f)^{\#}\, ds$ exists for almost all $(x, v)$ for any given $\psi \in CL$ and any $t \leqslant T$, and is measurable in $x, v$. Let $\chi_+$ be the characteristic function of the set in $(x, v)$ where the integral $I(x, v)$ is

$>0$. Consider first $\psi_+ = \chi_+ \psi$ (also in $CL$) and take an increasing sequence of characteristic functions $\chi_n \chi_+$, such that $f^\#$ is bounded on $\operatorname{supp} \chi_n \times [0, t]$. For the test function $\psi\chi_n$, (4.2) holds, and

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\Lambda\times\mathbf{R}^3} dx\, dv \int_0^t \chi_n \psi_+ Q(f, f)^\#\, ds \\
&\quad = \lim_{n\to\infty} \int_{\Lambda\times\mathbf{R}^3} dx\, dv\, \chi_n \int_0^t \psi_+ Q(f, f)^\#\, ds \\
&\quad = \int_{\Lambda\times\mathbf{R}^3} dx\, dv \int_0^t \psi_+ Q(f, f)^\#\, ds
\end{aligned}$$

We proceed analogously for $\psi - \psi_+$. The bounds on mass and energy follow from the corresponding ones for $f_n$ given by Theorem 5. The entropy bound follows in the limit from (4.17), since

$$\begin{aligned}
&\frac{\kappa}{2} \int_0^t \int_M (1-\chi_n)(ff_+)^\#\,(s)\, B\, ds\, d\mu \\
&\quad \leqslant \frac{\kappa}{2n^2} \int_0^t \int_M (v^2 + v_*^2)(ff_+)^\#\,(s)\, B\, ds\, d\mu \\
&\quad \leqslant \frac{\kappa}{2n^2} \int_{\Lambda\times\mathbf{R}^3} v^2 f_0\, dx\, dv \to 0 \qquad \text{when} \quad n\to\infty
\end{aligned} \tag{5.16}$$

Here (3.8) was used in the last step. Finally, the theorem for all $t \in R_+$ follows by a Cantor diagonalization argument.

*Proof of Lemma 10.* Set $f_R^n = f^n \wedge R$ and let $B_\lambda$ be the ball of radius $\lambda$ in $\mathbf{R}^3$. Choose $\lambda$ so large that $\operatorname{supp} \psi \subset \Lambda \times B_{\lambda/2} \times \mathbf{R}_+$ and denote by $\chi_\lambda$ the characteristic function of the set

$$((v, v_*) \in \mathbf{R}^3 \times \mathbf{R}^3;\, v^2 + v_*^2 \leqslant \lambda^2)$$

Evidently, for $n > \lambda$,

$$\begin{aligned}
&\int_{\Omega_{jn}} \left(\frac{(f^n f_-^n)'}{1+\delta f^n}\right)^\# \psi B \chi_n\, d\mu\, ds \\
&\quad \geqslant \int_{\Omega_{jn}} \left(\frac{(f^n f_-^n)'}{1+\delta f^n}\right)^\# \chi_\lambda \psi B\, d\mu\, ds \\
&\quad \geqslant \int_0^t ds \int_M ((f_R^n f_-^n)')^\# \,(1+\sqrt{\delta})^{-1}\, \psi B \chi_\lambda\, d\mu\, ds + \varepsilon' + \varepsilon''
\end{aligned} \tag{5.17}$$

where

$$|\varepsilon'| \leqslant C_{\lambda,R}\, o(\delta), \qquad |\varepsilon''| \leqslant C[l(j)]^{-1}$$

If $\{R_k\}_{k\in\mathbf{N}}$ is a sequence tending to infinity, and

$$\lim_{n'} f^{n'}_{R_k} = f_{R_k} \qquad \text{weakly in } L^1$$

then $\lim_{k\to\infty} f_{R_k} = f$ in $L^1(\Lambda\times\mathbf{R}^3\times(0,T))$. Using Lemma 8, choose $\{n'\}$ such that for a.a. $(x, v, u, s)\in\Lambda\times\mathbf{R}^3\times\mathscr{S}_+\times(0,T)$:

$$\lim_{n'} \int_{\mathbf{R}^3} f^{n'}_+ \psi^{\#'} B\chi_\lambda\, dv_* = \int_{\mathbf{R}^3} f_+ \psi^{\#'} B\chi_\lambda\, dv_* \tag{5.18}$$

(where $\#'$ denotes that $sv'$ is added to the $x$ variable).

Then

$$\begin{aligned}
&\lim_{n'} \int_0^t ds \int_M ((f^{n'}_R f^{n'}_-)')^{\#}\, (1+\sqrt{\delta})^{-1}\, \psi B\chi_\lambda\, d\mu \\
&\quad = \int_0^t ds \int_M ((f_R f_-)')^{\#}\, (1+\sqrt{\delta})^{-1}\, \psi B\chi_\lambda\, d\mu \\
&\quad \to \int_0^t ds \int_M ((f_R f_-)')^{\#}\, \psi B\chi_\lambda\, d\mu \qquad \text{as} \quad \delta\to 0+
\end{aligned} \tag{5.19}$$

From here Lemma 10 follows if we let $\lambda\to\infty$, $R_k\to\infty$, and $j\to\infty$ in this order. ▌

*Proof of Lemma 11.* With $\chi_\lambda$ as above, take $\lambda=\lambda(\delta)$ so that $\lim_{\delta\to 0+} [\delta\lambda(\delta)]^{-1}=0$. It is enough to take $n\geqslant\lambda$ and consider

$$I_n = \int_0^t \int_M \left(\frac{(f^n f^n_-)'}{1+\delta f^n}\right)^{\#} \chi_\lambda \psi B\chi_n\, d\mu\, ds \tag{5.20}$$

since

$$\int_0^T ds \int_M \left(\frac{(f^n f^n_-)'}{1+\delta f^n}\right)^{\#} (1-\chi_\lambda)\, \psi B\chi_n\, d\mu \leqslant \frac{C}{l(j)} + \frac{jC}{\delta\lambda(\delta)} \int v^2 f_0\, dx\, dv \tag{5.21}$$

For $\lambda$ fixed, outside some set in $\Lambda\times\mathbf{R}^3\times\mathscr{S}_+\times[0,T]$ of arbitrarily small measure $\eta>0$, for some subsequence $\{n'\}$, Eq. (5.18) holds with uniform convergence and bounded limit. Thus,

$$\lim_{n'} I_{n'} \leqslant \int_0^t ds \int_M (f'f'_-)^{\#}\, \psi B\chi_{\lambda(\delta)}\, d\mu + \frac{C}{l(j)} + C_{j,\delta}\, o(\eta) \tag{5.22}$$

The lemma follows if we take $\eta\to 0+$, $\delta\to 0+$, and $j\to\infty$ in this order. ▌

**Theorem 12.** Let $f_0$ satisfy (2.5). Then there exists a function $f \in C([0, \infty), L^1(\Lambda \times \mathbf{R}^3))$, satisfying the EE (2.1) (in any of the four equivalent forms of Lemma 1) with initial value $f_0$. For $t > 0$ mass and energy are bounded by their initial values and the entropy satisfies

$$\int_{\Lambda \times \mathbf{R}^3} f(t) \log f(t)\, dx\, dv$$
$$\leqslant \int_{\Lambda \times \mathbf{R}^3} f_0(\log f_0 + v^2)\, dx\, dv + \frac{\kappa\sigma^2}{2}\left(\int_{\Lambda \times \mathbf{R}} f_0\, dx\, dv\right)^2 \quad (5.23)$$

Finally, (5.9) (with $\varepsilon = 0$) holds for $T > 0$.

*Proof.* Start from a sequence of solutions $f_{\varepsilon_\nu}$ of Eq. (5.11) with $\varepsilon_\nu \to 0$ when $\nu \to \infty$, all with initial value $f_{\varepsilon_\nu}(0) = f_0$. The limiting behavior of Eq. (4.19) will be needed. But essentially by Lemma 8 this limit gives

$$\int_{O \times \mathscr{S}_+} \frac{\kappa}{2} f'_{\varepsilon_\nu} f'_{\varepsilon_\nu -} B\, ds\, d\mu$$
$$\leqslant j \int_{O \times \mathscr{S}_+} \frac{\kappa}{2} f_{\varepsilon_\nu} f_{\varepsilon_\nu -} B\, ds\, d\mu$$
$$+ \frac{1}{l(j)} \left\{ \int_{\Lambda \times \mathbf{R}^3} f_0(\log^+ f_0 + 2v^2)\, dx\, dv + \frac{\kappa\sigma^2}{2}\left(\int_{\Lambda \times \mathbf{R}^2} f_0\, dx\, dv\right)^2\right.$$
$$\left. + \int_{\mathbf{R}^3} \exp(-v^2 - 1)\, dv + \frac{\pi\varepsilon_\nu t\sigma^2}{3e}\left[\int_{\mathbf{R}^3} (|v| + 1) \exp(-v^2)\, dv\right]^2\right\}$$
$$(5.24)$$

Using (5.24) instead of (4.19), together with Theorem 9 for the other properties of $f_{\varepsilon_\nu}$, the proof of Theorem 12 almost line by line follows the proof of Theorem 9 and is left to the reader. ∎

## 6. CONVERGENCE TO A SOLUTION OF THE BOLTZMANN EQUATION

The type of argument used in the previous section implies that the solutions of the EE provided by Theorem 12 converge to solutions of the BE when the diameter $\sigma$ tends to zero and we assume that, with a suitable rescaling, $\sigma^2\kappa_\sigma$ remains constant in the process. We follow again the line of argument of ref. 7.

**Theorem 13.** For any sequence $\{\sigma_j\}_{j \in \mathbf{N}}$ of reals with $\lim_{j \to \infty} \sigma_j = 0$, and any corresponding sequence of solutions $\{f_{\sigma_j}\}_{j \in \mathbf{N}}$ of Theorem 12, there is a subsequence $\{\sigma_{j'}\}$ for which $\{f_{\sigma_{j'}}\}$ converges weakly in $L^1$ to a solution $f$ of the BE for elastic spheres.

For the necessary changes in the proof of Theorem 12 to obtain this result, the following lemma is required.

**Lemma 14.** Assume that $g^n$ tends to $g$ weakly in $L^1(\Lambda \times \mathbf{R}^3 \times (0, T))$, $\lim \sigma_n = 0+$, and that for all $\psi \in L^\infty(M \times (0, T))$ with bounded support

$$\lim_{n \to \infty} \int_{\mathbf{R}^3} g^n \psi \, dv_* = \int_{\mathbf{R}^3} g\psi \, dv_* \qquad \text{in} \quad L^1(\Lambda \times \mathbf{R}^3 \times \mathscr{S}_+ \times (0, T)) \tag{6.1}$$

Then, strongly in $L^1(\Lambda \times \mathbf{R}^3 \times \mathscr{S}_+ \times (0, T))$,

$$\lim_{n \to \infty} \int_{\mathbf{R}^3} g^n(x \pm \sigma_n u, v_*, s)\, \psi(x, v, v_*, u, s)\, dv_*$$
$$= \int_{\mathbf{R}^3} g(x, v_*, s)\, \psi(x, v, v_*, u, s)\, dv_* \tag{6.2}$$

*Proof.* The lemma follows if

$$\lim_{n \to \infty} \int_{\mathbf{R}^3} g^n(x, v_*, s)\, \psi(x \pm \sigma_n u, v, v_*, u, s)\, dv_*$$
$$= \int_{\mathbf{R}^3} g(x, v_*, s)\, \psi(x, v, v_*, u, s)\, dv_* \tag{6.3}$$

But this holds for continuous $\psi$, hence by approximation for those indicated in the statement of the lemma. ∎

Recall that $f_{\sigma_j} = \lim_{n \to \infty} f^n_{\sigma_j}$ for some sequence of truncated solutions $\{f^n_{\sigma_j}\}_{n \in \mathbf{N}}$ with a truncation tending to infinity with $n$, and let $f$ be the weak $L^1$ limit of a subsequence of $\{f_{\sigma_j}\}_{j \in \mathbf{N}}$. It follows that there is a sequence of diameters $\{\sigma_{j'}\}$ and a corresponding sequence of functions $\{f^{n_{j'}}_{\sigma_{j'}}\}$, with the latter converging weakly in $L^1$ to $f$. Denote the latter sequence by $\{f^j\}_{j \in \mathbf{N}}$ and the subsequence of radii by $\{\sigma_j\}_{j \in \mathbf{N}}$. Notice that the inequality relating the gain and loss terms, Eq. (5.24), holds for $f^j$, $j \in \mathbf{N}$.

Lemma 14 now implies the following variant of Lemma 8.

**Lemma 8′.** There is a subsequence $\{j'\}$ of $\mathbf{N}$ such that

$$\lim_{j'} \int_{\mathbf{R}^3} f^{j'}(x \pm \sigma_{j'} u, v_*, t)\, B\psi \, dv_*$$
$$= \int_{\mathbf{R}^3} f(x, v_*, t)\, B\psi \, dv_*$$
$$\text{in} \quad L^1_{\text{loc}}(\Lambda \times \mathbf{R}^3 \times \mathscr{S}_+ \times (0, T)) \qquad \text{for} \quad \psi \in L^\infty(M \times (0, T)) \tag{6.4}$$

*Proof of Theorem 14.* The proof follows the argument used for Theorem 12 with obvious replacement of Lemma 8 by Lemma 8′ and a few references to Lemma 14. ∎

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